

HILBERT FUNCTIONS AND SET OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT. In this paper we study the problem of classifying the Hilbert functions of zero-dimensional schemes in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, in the main result of the paper we give conditions to determine some Hilbert functions of set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and we describe geometrically these schemes. Moreover, we show that the Hilbert functions of these schemes depend only on the distribution of the points on a set of $(1, 0)$ and $(0, 1)$ -lines.

1. INTRODUCTION

Given $Q = \mathbb{P}^1 \times \mathbb{P}^1$, Giuffrida, Maggioni and Ragusa in [2] have investigated zero-dimensional schemes in Q , studying in particular their Hilbert functions, which turn out to be matrices of integers with infinite entries and with particular numerical properties. These numerical conditions are sufficient to characterize the Hilbert functions of arithmetically Cohen-Macaulay zero-dimensional schemes in Q (see [2]) and by the Hilbert function of an arithmetically Cohen-Macaulay zero-dimensional scheme it is possible to determine a geometrical description of the scheme. Other results about the Hilbert functions of zero-dimensional schemes in Q have been obtained for fat points (see [3], [4], [5], [6] and [9]). In this paper in Theorem 6 we give numerical conditions to determine Hilbert functions of some set of points in Q . In particular we describe these schemes and we show that any zero-dimensional scheme having in a grid of $(1, 0)$ and $(0, 1)$ -lines the same configuration of points has the same Hilbert function.

Given a zero-dimensional scheme $X \subset Q$ and a point $P \in X$, in Section 3 we look for the Hilbert function of $X \setminus \{P\}$ in relation to the Hilbert function of X , giving a sufficient condition in Corollary 1. In particular, we show that under this condition there exists just one separator for $P \in X$ and it has minimal degree (see [7] and [8]). As a consequence we can partially improve some results given in [1] on the Hilbert function of the union of a zero-dimensional scheme X with a particular set of points of Q .

In Section 5 we prove Theorem 6, in which we give sufficient conditions to determine some Hilbert functions of set of points in Q . The conditions in Theorem 6 are quite technical, but they show a way to new conditions for a characterization of Hilbert functions of zero-dimensional schemes in Q .

In Example 1 we give a matrix satisfying some of the conditions Theorem 6 and an application of Theorem 6 is given in Example 2, while in Example 3 we show that the conditions of Theorem 6 are not necessary.

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2. NOTATION

Let k be an algebraically closed field, let $\mathbb{P}^1 = \mathbb{P}_k^1$, let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let \mathcal{O}_Q be its structure sheaf. Let us consider the bi-graded ring $S = H_*^0 \mathcal{O}_Q = \bigoplus_{a,b \geq 0} H^0 \mathcal{O}_Q(a,b)$. For any sheaf \mathcal{F} and any $a, b \in \mathbb{Z}$ we define $\mathcal{F}(a,b) = \mathcal{F} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(a,b)$.

For any bi-graded S -module N let $N_{i,j}$ be the component of degree (i,j) . For any $(i_1, j_1), (i_2, j_2) \in \mathbb{N}^2$ we write $(i_1, j_1) \geq (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \geq j_2$. Given a 0-dimensional scheme $X \subset Q$, let $I(X) \subset S$ be the associated saturated ideal and $S(X) = S/I(X)$ the associated graded ring.

Definition 1. The function $M_X: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$ defined by:

$$M_X(i, j) = \dim_k S(X)_{i,j} = (i+1)(j+1) - \dim_k I(X)_{i,j}$$

is called the *Hilbert function* of X . The function M_X can be represented as an infinite matrix with integer entries $M_X = (M_X(i, j)) = (m_{ij})$ called *Hilbert matrix* of X .

In this paper we denote $M_X(i, j)$ also by $M_X^{(i,j)}$ to simplify the notation. Note that $M_X(i, j) = 0$ for either $i < 0$ or $j < 0$, so we restrict ourselves to the range $i \geq 0$ and $j \geq 0$. Moreover, for $i \gg 0$ and $j \gg 0$ $M_X(i, j) = \deg X$.

Definition 2. Given the Hilbert matrix M_X of a zero-dimensional scheme $X \subset Q$, the *first difference of the Hilbert function* of X is the matrix $\Delta M_X = (c_{ij})$, where $c_{ij} = m_{ij} - m_{i-1,j} - m_{i,j-1} + m_{i-1,j-1}$.

We consider the matrices $\Delta^R M_X = (a_{ij})$ and $\Delta^C M_X = (b_{ij})$, with $a_{ij} = m_{ij} - m_{i,j-1}$ and $b_{ij} = m_{ij} - m_{i-1,j}$. Note that for any $i, j \geq 0$:

$$(1) \quad a_{ij} = \sum_{t=0}^i c_{tj} \quad \text{and} \quad b_{ij} = \sum_{t=0}^j c_{it}.$$

For any matrix M with infinite entries it is possible to define in a similar way ΔM , $\Delta^R M$ and $\Delta^C M$.

Definition 3 ([2, Definition 2.2]). Let $M = (m_{ij})$ be a matrix such that $m_{ij} = 0$ for $i < 0$ and $j < 0$. We say that M is admissible if $\Delta M = (c_{ij})$ satisfies the following conditions:

- (1) $c_{ij} \leq 1$ and $c_{ij} = 0$ for $i \gg 0$ or $j \gg 0$;
- (2) if $c_{ij} \leq 0$, then $c_{rs} \leq 0$ for any $(r, s) \geq (i, j)$;
- (3) for every (i, j) $0 \leq \sum_{t=0}^j c_{it} \leq \sum_{t=0}^j c_{i-1,t}$ and $0 \leq \sum_{t=0}^i c_{tj} \leq \sum_{t=0}^i c_{t,j-1}$.

Theorem 1 ([2, Theorem 2.11]). *If $X \subset Q$ is a 0-dimensional scheme, then M_X is an admissible matrix.*

If $X \subset Q$ is a zero-dimensional scheme, then $2 \leq \text{depth } S(X) \leq 3$.

Definition 4. A zero-dimensional scheme $X \subset Q$ is called arithmetically Cohen-Macaulay (ACM) if $\text{depth } S(X) = 2$.

Theorem 2 ([2, Theorem 4.1]). *A zero-dimensional scheme $X \subset Q$ is ACM if and only if $c_{ij} \geq 0$ for any (i, j) .*

Given an admissible matrix M , we define:

$$(2) \quad T = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid c_{ij} < 0\}.$$

Then for any $(i, j) \in T$ we set:

$$(3) \quad I_{ij} = \{0, \dots, -c_{ij} - 1\}.$$

Remark 1. If $X \subset Q$ is a 0-dimensional scheme, let us consider $a = \min\{i \in \mathbb{N} \mid I(X)_{i,0} \neq 0\} - 1$ and $b = \min\{j \in \mathbb{N} \mid I(X)_{0,j} \neq 0\} - 1$. Then by Theorem 1 ΔM_X is zero out of the rectangle with opposite vertices $(0, 0)$ and (a, b) , because $c_{a+1,0} = c_{0,b+1} = 0$. In this case we say that ΔM_X is of size (a, b) .

Let $X \subset Q$ be a zero-dimensional scheme and let L be a line defined by a form l . Let $J = (I(X), l)$ and let $d = \deg(\text{sat } J)$. Then we call d the number of points of X on the line L and, by abuse of notation, we define $d = \#(X \cap L)$. We say that L is disjoint from X if $d = 0$.

For any $i \geq 0$ we set $j(i) = \min\{t \in \mathbb{N} \mid m_{it} = m_{it+1}\}$ and similarly for any $j \geq 0$ we set $i(j) = \min\{t \in \mathbb{N} \mid m_{tj} = m_{t+1j}\}$.

Theorem 3 ([2, Theorem 2.12]). *Let $X \subset Q$ be a zero-dimensional scheme and let $M_X = (m_{ij})$ be its Hilbert matrix. Then for every $j \geq 0$ there are just $a_{i(0)j} - a_{i(0)j+1}$ lines of type $(1, 0)$ each containing just $j + 1$ points of X and, similarly, for every $i \geq 0$ there are just $b_{ij(0)} - b_{i+1j(0)}$ lines of type $(0, 1)$ each containing just $i + 1$ points of X .*

Now we recall the following definition:

Definition 5. Let $X \subset Q$ be a zero-dimensional scheme and let $P \in X$. The multiplicity of X in P , denoted by $m_X(P)$, is the length of $\mathcal{O}_{X,P}$.

Given $P \in Q$, we denote by I_P the maximal ideal of S associated to P . If $X \subset Q$ is a 0-dimensional scheme, then $I(X) = \cap_{P' \in X} J_{P'}$ for some ideal $J_{P'}$ such that $\sqrt{J_{P'}} = I_{P'}$.

Definition 6. Given a zero-dimensional scheme $X \subset Q$ and $P \in X$ such that $m_X(P) = 1$, we say that $f \in S$ is a *separator* for $P \in X$ if $f(P) \neq 0$ and $f \in \cap_{P' \in X \setminus \{P\}} J_{P'}$.

This definition generalizes the definition of a separator for a point in a reduced zero-dimensional scheme in a multiprojective space given by [6].

3. SEPARATORS AND HILBERT FUNCTIONS

Let $X \subset Q$ be a zero-dimensional scheme and let M_X be its Hilbert matrix. In all this paper we suppose that ΔM_X is of size (a, b) and we denote by R_0, \dots, R_a and C_0, \dots, C_b , respectively, the $(1, 0)$ and $(0, 1)$ -lines containing X and each one at least one point of X .

Theorem 4. *Let $P = R_h \cap C_k \in X$ for some $h \in \{0, \dots, a\}$ and $k \in \{0, \dots, b\}$ and suppose that $m_X(P) = 1$. Let $Z = X \setminus \{P\}$, $p = \#(Z \cap R_h)$ and $q = \#(Z \cap C_k)$. If there exists a separator in degree (q, p) for $P \in X$, then:*

$$\Delta M_Z^{(i,j)} = \begin{cases} \Delta M_X^{(i,j)} & \text{if } (i, j) \neq (q, p) \\ \Delta M_X^{(i,j)} - 1 & \text{if } (i, j) = (q, p). \end{cases}$$

Proof. It is easy to see that $\Delta M_Z^{(i,j)} = \Delta M_X^{(i,j)}$ for any (i,j) with either $i < q$ or $j < p$. Indeed, taken (i,j) with $i < q$ any (i,j) -curve containing Z must contain C_k and so $h^0 \mathcal{I}_Z(i,j) = h^0 \mathcal{I}_X(i,j)$ and $\Delta M_Z^{(i,j)} = \Delta M_X^{(i,j)}$. The proof works in a similar way if $j < p$.

By the exact sequence:

$$(4) \quad 0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_P \rightarrow 0$$

we see that $h^0 \mathcal{I}_Z(q,p) > h^0 \mathcal{I}_X(q,p)$ if and only if $h^0 \mathcal{I}_Z(q,p) = h^0 \mathcal{I}_X(q,p) + 1$. This means that it must be:

$$\Delta M_Z^{(q,p)} = \Delta M_X^{(q,p)} - 1.$$

Now we only need to prove that $\Delta M_Z^{(i,j)} = \Delta M_X^{(i,j)}$ for any $(i,j) > (q,p)$. By (4) we see that for any (i,j) :

$$(5) \quad h^0 \mathcal{I}_X(i,j) \leq h^0 \mathcal{I}_Z(i,j) \leq h^0 \mathcal{I}_X(i,j) + 1$$

which is equivalent to:

$$M_X^{(i,j)} - 1 \leq M_Z^{(i,j)} \leq M_X^{(i,j)}.$$

Since $h^0 \mathcal{I}_Z(q,p) = h^0 \mathcal{I}_X(q,p) + 1$, by (5) we see that it must be $h^0 \mathcal{I}_Z(i,j) = h^0 \mathcal{I}_X(i,j) + 1$ for any $(i,j) \geq (q,p)$. In particular this means that $M_Z^{(i,j)} = M_X^{(i,j)} - 1$ for any $(i,j) \geq (q,p)$. Now the conclusion follows easily. \square

Theorem 5. *Let $P = R_h \cap C_k \in X$ for some $h \in \{0, \dots, a\}$ and $k \in \{0, \dots, b\}$ such that $m_X(P) = 1$ and let $p+1 = \#(X \cap R_h)$ and $q+1 = \#(X \cap C_k)$. Suppose that one of the following conditions holds:*

- (1) $p = b$;
- (2) $q = a$;
- (3) $p < b$, $q < a$ and $\Delta M_X^{(i,j)} = 0$ for any $(i,j) \geq (q+1, p+1)$.

Then there exists a separator for $P \in X$ in degree (q,p) .

Proof. We divide the proof in different steps. Let $Z = X \setminus \{P\}$.

Step 1. There exists \bar{j} with $p \leq \bar{j} \leq b$ such that one the following conditions holds:

- (1) $\Delta M_Z^{(q,j)} = \Delta M_X^{(q,j)}$ for any $j < \bar{j}$ and $\Delta M_Z^{(q,\bar{j})} < \Delta M_X^{(q,\bar{j})}$;
- (2) $\Delta M_Z^{(q,j)} = \Delta M_X^{(q,j)}$ for any $p \leq j \leq b$.

Since $Z \subset X$ we see that $M_Z^{(q,p)} \leq M_X^{(q,p)}$. Moreover, as we have seen in the proof of Theorem 4 $M_Z^{(i,j)} = M_X^{(i,j)}$ for any $i < q$ or $j < p$. This implies that $\Delta M_Z^{(q,p)} \leq \Delta M_X^{(q,p)}$. If $\Delta M_Z^{(q,p)} = \Delta M_X^{(q,p)}$, then we can repeat the previous procedure to show that $\Delta M_Z^{(q,p+1)} \leq \Delta M_X^{(q,p+1)}$. By iterating this procedure we get the conclusion of Step 1.

Step 2. The following equalities hold:

- (1) $\sum_{j=p}^b \Delta M_Z^{(q,j)} = \sum_{j=p}^b \Delta M_X^{(q,j)} - 1$;
- (2) for any $i \in \{q+1, \dots, a\}$ $\sum_{j=p}^b \Delta M_Z^{(i,j)} = \sum_{j=p}^b \Delta M_X^{(i,j)}$.

Let us first note that by Theorem 3:

$$b_{q-1j(0)}(Z) - b_{qj(0)}(Z) = \sum_{j \leq b} \Delta M_Z^{(q-1,j)} - \sum_{j \leq b} \Delta M_Z^{(q,j)}$$

is equal to the number of $(0, 1)$ -lines containing precisely q points of Z , while:

$$b_{q-1j(0)}(X) - b_{qj(0)}(X) = \sum_{j \leq b} \Delta M_X^{(q-1,j)} - \sum_{j \leq b} \Delta M_X^{(q,j)}$$

is equal to the number of $(0, 1)$ -lines containing precisely q points of X . By hypothesis it must be:

$$b_{q-1j(0)}(Z) - b_{qj(0)}(Z) = \sum_{j \leq b} \Delta M_Z^{(q-1,j)} - \sum_{j \leq b} \Delta M_Z^{(q,j)} = \sum_{j \leq b} \Delta M_X^{(q-1,j)} - \sum_{j \leq b} \Delta M_X^{(q,j)} + 1$$

Since $h^0 \mathcal{I}_Z(i, j) = h^0 \mathcal{I}_X(i, j)$ for any $i < q$ or $j < p$, this implies that:

$$(6) \quad \sum_{j \leq b} \Delta M_Z^{(q,j)} = \sum_{j \leq b} \Delta M_X^{(q,j)} - 1.$$

In a similar way we see that:

$$b_{qj(0)}(Z) - b_{q+1j(0)}(Z) = \sum_{j \leq b} \Delta M_Z^{(q,j)} - \sum_{j \leq b} \Delta M_Z^{(q+1,j)} = \sum_{j \leq b} \Delta M_X^{(q,j)} - \sum_{j \leq b} \Delta M_X^{(q+1,j)} - 1$$

which implies by (6) that $\sum_{j \leq b} \Delta M_Z^{(q+1,j)} = \sum_{j \leq b} \Delta M_X^{(q+1,j)}$.

Let us now suppose that for some $i \geq q + 1$, with $i < a$, we have:

$$(7) \quad \sum_{j \leq b} \Delta M_Z^{(i,j)} = \sum_{i \leq b} \Delta M_X^{(i,j)}.$$

We will show that:

$$(8) \quad \sum_{j \leq b} \Delta M_Z^{(i+1,j)} = \sum_{j \leq b} \Delta M_X^{(i+1,j)}.$$

Again, by Theorem 3 $\sum_{j \leq b} \Delta M_Z^{(i,j)} - \sum_{j \leq b} \Delta M_Z^{(i+1,j)}$ is equal to the number of $(0, 1)$ -lines containing precisely $i+1$ points of Z , while $\sum_{j \leq b} \Delta M_X^{(i,j)} - \sum_{j \leq b} \Delta M_X^{(i+1,j)}$ is equal to the number of $(0, 1)$ -lines containing precisely $i+1$ points of X . By hypothesis it must be:

$$\sum_{j \leq b} \Delta M_Z^{(i,j)} - \sum_{j \leq b} \Delta M_Z^{(i+1,j)} = \sum_{j \leq b} \Delta M_X^{(i,j)} - \sum_{j \leq b} \Delta M_X^{(i+1,j)}.$$

By (7) it means that (8) holds, so that $\sum_{j \leq b} \Delta M_Z^{(i,j)} = \sum_{j \leq b} \Delta M_X^{(i,j)}$ for any i with $q+1 \leq i \leq a$.

The statement of the theorem is proved if we show the following:

Step 3. $h^0 \mathcal{I}_Z(q, p) = h^0 \mathcal{I}_X(q, p) + 1$.

In the cases $p = b$ and $q = a$ by Step 2 we easily get Step 3. So from now on we suppose that $p < b$ and $q < a$.

By Step 1 and Step 2 we see that there exists \bar{j} with $p \leq \bar{j} \leq b$ such that $\Delta M_Z^{(q,j)} = \Delta M_X^{(q,j)}$ for any $j < \bar{j}$ and $\Delta M_Z^{(q,\bar{j})} < \Delta M_X^{(q,\bar{j})}$. Let us suppose that $\bar{j} \geq p+1$. Then $\Delta M_Z^{(q,\bar{j})} \leq 0$ and by Theorem 1 we see that $\Delta M_Z^{(i,\bar{j})} \leq 0$ for any $i \geq q$. By Step 2 and by hypothesis we see that:

$$\sum_{i=q+1}^a \Delta M_Z^{(i,\bar{j})} = \Delta M_X^{(q,\bar{j})} - \Delta M_Z^{(q,\bar{j})} > 0.$$

So $\sum_{i=q+1}^a \Delta M_Z^{(i, \bar{j})} > 0$, but this contradicts that fact that $\Delta M_Z^{(i, \bar{j})} \leq 0$ for any $i \geq q$. This means that $\bar{j} = p$, i.e. $\Delta M_Z^{(q, p)} < \Delta M_X^{(q, p)}$. Since, as we have seen, $h^0 \mathcal{I}_Z(i, j) = h^0 \mathcal{I}_X(i, j)$ for any $(i, j) < (q, p)$, it gives us the inequality $M_Z^{(q, p)} < M_X^{(q, p)}$, which means that $h^0 \mathcal{I}_Z(q, p) > h^0 \mathcal{I}_X(q, p)$. But by the exact sequence:

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_P \rightarrow 0$$

we see that $h^0 \mathcal{I}_Z(q, p) > h^0 \mathcal{I}_X(q, p)$ if and only if $h^0 \mathcal{I}_Z(q, p) = h^0 \mathcal{I}_X(q, p) + 1$ and the statement is proved. \square

Corollary 1. *Let $P = R_h \cap C_k \in X$ for some $h \in \{0, \dots, a\}$ and $k \in \{0, \dots, b\}$ such that $m_X(P) = 1$. Given $Z = X \setminus \{P\}$, $p = \#(Z \cap R_h)$ and $q = \#(Z \cap C_k)$, suppose that one of the following conditions holds:*

- (1) $p = b$;
- (2) $q = a$;
- (3) $p < b$, $q < a$ and $\Delta M_X^{(i, j)} = 0$ for any $(i, j) \geq (q+1, p+1)$.

Then:

$$\Delta M_Z^{(i, j)} = \begin{cases} \Delta M_X^{(i, j)} & \text{if } (i, j) \neq (q, p) \\ \Delta M_X^{(i-1, j)} - 1 & \text{if } (i, j) = (q, p). \end{cases}$$

Proof. The proof follows by Theorem 4 and Theorem 5. \square

Corollary 2. *Let X be an ACM zero-dimensional scheme and let $P = R_h \cap C_k \in X$ for some $h \in \{0, \dots, a\}$ and $k \in \{0, \dots, b\}$ such that $m_X(P) = 1$. Given $Z = X \setminus \{P\}$, $p = \#(Z \cap R_h)$ and $q = \#(Z \cap C_k)$, we have:*

$$\Delta M_Z^{(i, j)} = \begin{cases} \Delta M_X^{(i, j)} & \text{if } (i, j) \neq (q, p) \\ \Delta M_X^{(i-1, j)} - 1 & \text{if } (i, j) = (q, p). \end{cases}$$

Proof. By [1, Proposition 4.1] we see that $\Delta M_X^{(i, j)} = 0$ for any $(i, j) \geq (q+1, p+1)$. Then the conclusion follows by Corollary 1. \square

In the following we slightly improve the result given in [1, Theorem 3.1].

Corollary 3. *Let R be a $(1, 0)$ -line disjoint from X . Let C_{b+1}, \dots, C_n , $n \geq b$, be arbitrary $(0, 1)$ -lines and $i_1, \dots, i_r \in \{0, \dots, b\}$. Let $\mathcal{P} = \{R \cap C_i \mid i \in \{0, \dots, n\}, i \neq i_1, \dots, i_r\}$ and let $W = X \cup \mathcal{P}$. Suppose also that on the $(0, 1)$ -line C_{i_k} there are q_k points of X for $k = 1, \dots, r$ and that $q_1 \leq q_2 \leq \dots \leq q_r$. Then, given $T = \{(q_1, n), (q_2, n-1), \dots, (q_r, n-r+1)\}$, we have:*

$$\Delta M_W^{(i, j)} = \begin{cases} 1 & \text{if } i = 0, j \leq n \\ 0 & \text{if } i = 0, j \geq n+1 \\ \Delta M_X^{(i-1, j)} & \text{if } i \geq 1 \text{ and } (i, j) \notin T \\ \Delta M_X^{(i-1, j)} - 1 & \text{if } i \geq 1 \text{ and } (i, j) \in T \end{cases}$$

if one of the following conditions holds:

- (1) $r = 1$;
- (2) $r \geq 2$ and for any $k \in \{2, \dots, r\}$ and $i \geq q_k$ $\Delta M_X^{(i, n-k+2)} = 0$.

Proof. Let $Y = X \cup (R \cap (C_0 \cup \dots \cup C_n))$. Then the statement follows by [2, Lemma 2.15] and by Corollary 1. \square

4. TECHNICAL RESULTS

In this section we prove some technical results that will be useful in the proof of Theorem 6. In all this section we denote by M an admissible matrix and we keep the notation given previously.

Proposition 1. *Let us suppose that for some (i_1, j_1) and (i_2, j_2) with $j_1 > j_2$ the following conditions hold:*

- (1) $c_{i_1 j_1} < 0$ and $c_{i_2 j_2} \leq 0$;
- (2) $a_{i_1 j_1} + r \geq a_{i_2 j_2}$, for some $r \in I_{i_1 j_1}$.

Then $i_1 \leq i_2$.

Proof. Let us suppose that $i_1 > i_2$. Then by hypothesis we have $\sum_{t=0}^{i_1} c_{t j_1} + r \geq \sum_{t=0}^{i_2} c_{t j_2}$ and so by Theorem 1:

$$0 \geq \sum_{t=0}^{i_2} c_{t j_1} - \sum_{t=0}^{i_2} c_{t j_2} \geq -r - \sum_{t=i_2+1}^{i_1} c_{t j_1}.$$

This implies that:

$$0 \leq r + \sum_{t=i_2+1}^{i_1} c_{t j_1} \leq r + c_{i_1 j_1} < 0,$$

by hypothesis and by the fact that by Theorem 1 $c_{t j_1} \leq 0$ for any $t \geq i_2$. \square

In a similar way it is possible to prove the following:

Proposition 2. *Let us suppose that for some (i_1, j_1) and (i_2, j_2) with $i_1 > i_2$ the following conditions hold:*

- (1) $c_{i_1 j_1} < 0$ and $c_{i_2 j_2} \leq 0$;
- (2) $b_{i_1 j_1} + r \geq b_{i_2 j_2}$, for some $r \in I_{i_1 j_1}$.

Then $j_1 \leq j_2$.

Another technical result is:

Proposition 3. *Let us suppose that for some (i_1, j_1) and (i_2, j_2) , with $j_2 < j_1 - 1$, the following conditions hold:*

- (1) $c_{i_1 j_1} < 0$ and $c_{i_2 j_2} \leq 0$;
- (2) $a_{i_1 j_1} + r \geq a_{i_2 j_2}$, for some $r \in I_{i_1 j_1}$.

Then there exists (i, j) with $j_2 < j < j_1$ and $i \leq i_2$ such that $c_{ij} < 0$ and $a_{i_1 j_1} + r + c_{ij} + 1 \leq a_{ij} \leq a_{i_1 j_1} + r$.

Proof. First note that by Proposition 1 it must be $i_1 \leq i_2$. Suppose that for every (i, j) with $j_2 < j < j_1$ and $i \leq i_2$ we have $c_{ij} \geq 0$. Then this implies that $a_{i_2 j_2+1} \geq a_{i_1-1 j_2+1}$, by which we get:

$$a_{i_2 j_2} \geq a_{i_2 j_2+1} \geq a_{i_1-1 j_2+1} \geq a_{i_1-1 j_1}.$$

However:

$$a_{i_2 j_2} \leq a_{i_1 j_1} + r = a_{i_1-1 j_1} + c_{i_1 j_1} + r < a_{i_1-1 j_1},$$

which gives us a contradiction.

Take j with $j_2 < j < j_1$ such that $c_{ij} < 0$ for some $i \leq i_2$. Then we can choose i in such a way that $a_{ij} = a_{i_2 j}$. Then by Theorem 1 we see that $a_{ij} \leq a_{i_2 j_2} \leq a_{i_1 j_1} + r$. If $a_{i_1 j_1} + r + c_{ij} + 1 \leq a_{ij}$, then we get the conclusion. So we can suppose that:

$$(9) \quad a_{ij} < a_{i_1 j_1} + r + c_{ij} + 1.$$

Take $i' < i$ such that $c_{i'j} < 0$ and $c_{kj} = 0$ for $k = i'+1, \dots, i-1$. Then $a_{ij} = c_{ij} + a_{i'j}$ and (9) is equivalent to:

$$a_{i'j} \leq a_{i_1j_1} + r.$$

Again, if $a_{i_1j_1} + r + c_{i'j} + 1 \leq a_{i'j}$, then the conclusion follows. Otherwise we proceed as before. Iterating this procedure we see that either we get the conclusion or $a_{kj} \leq a_{i_1j_1} + r$ for k such that $c_{kj} = 1$ and $c_{k+1j} \leq 0$. So we can suppose that such a k exists. Then we see that $a_{kj} = \max\{a_{ij} \mid i \geq 0\} \geq a_{i_1-1j}$, so that $a_{i_1-1j} \leq a_{i_1j_1} + r < a_{i_1-1j_1}$. But by Theorem 1 this is not possible. \square

In a similar way it is possible to prove the following:

Proposition 4. *Let us suppose that for some (i_1, j_1) and (i_2, j_2) , with $i_2 < i_1 - 1$, the following conditions hold:*

- (1) $c_{i_1j_1} < 0$ and $c_{i_2j_2} \leq 0$;
- (2) $b_{i_1j_1} + r \geq b_{i_2j_2}$, for some $r \in I_{i_1j_1}$.

Then there exists (i, j) with $i_2 < i < i_1$ and $j \leq j_2$ such that $c_{ij} < 0$ and $b_{i_1j_1} + r + c_{ij} + 1 \leq b_{ij} \leq b_{i_1j_1} + r$.

Remark 2. By Proposition 3 it follows that, given $(i_1, j_1), (i_2, j_2) \in T$, $r_1 \in I_{i_1j_1}$ and $r_2 \in I_{i_2j_2}$ such that $a_{i_1j_1} + r_1 = a_{i_2j_2} + r_2$, for any j with $j_1 \leq j \leq j_2$ there exists i such that $(i, j) \in T$ and $a_{ij} + r = a_{i_1j_1} + r_1 = a_{i_2j_2} + r_2$ for some $r \in I_{ij}$.

Of course, a similar result follows by Proposition 4.

Now we prove a result on $\Delta^R M$.

Proposition 5. *Let $(i_1, j_1), (i_2, j_1) \in T$ with $i_2 < i_1$. Then $a_{i_2j_1} + s > a_{i_1j_1} + r$, for any $r \in I_{i_1j_1}$ and $s \in I_{i_2j_1}$.*

Proof. Let us suppose that $a_{i_2j_1} + s \leq a_{i_1j_1} + r$. Note that $a_{i_1j_1} = a_{i_2j_1} + \sum_{i=i_2+1}^{i_1} c_{ij_1}$. Then we have:

$$(10) \quad s \leq \sum_{i=i_2+1}^{i_1} c_{ij_1} + r.$$

However $c_{i_1j_1} + r < 0$ and by Theorem 1 $c_{ij_1} \leq 0$ for any $i > i_2$. Then by (10) we get $s < 0$, which gives us a contradiction. \square

In a similar way it is possible to prove the following:

Proposition 6. *Let $(i_1, j_1), (i_1, j_2) \in T$ with $j_2 < j_1$. Then $b_{i_1j_2} + s > b_{i_1j_1} + r$, for any $r \in I_{i_1j_1}$ and $s \in I_{i_1j_2}$.*

Given the admissible matrix M of size (a, b) , let us consider R_0, \dots, R_a and C_0, \dots, C_b pairwise distinct arbitrary $(1, 0)$ and $(0, 1)$ -lines. Let $P_{ij} = R_i \cap C_j$ and let us consider the following reduced ACM zero-dimensional scheme:

$$X = \{P_{ij} \mid c_{ij} = 1\}.$$

Under this notation we prove the following:

Proposition 7. *Let $p \in \mathbb{N}$ such that:*

$$\{(i, j) \in T \mid p + c_{ij} + 1 \leq a_{ij} \leq p\} \neq \emptyset$$

and let:

$$k = \max\{j \mid \exists (i, j) \in T, p + c_{ij} + 1 \leq a_{ij} \leq p\}.$$

Then $0 \leq p \leq a$ and $\#(X \cap R_p) = k + 1$.

Proof. Let $(h, k) \in T$ such that $p + c_{hk} + 1 \leq a_{hk} \leq p$. Then there exists $s \in I_{hk}$ such that $a_{hk} + s = p$. This implies that $0 \leq p \leq a$.

Now we prove that $\#(X \cap R_p) = k + 1$. We will show that $c_{pk} = 1$ and $c_{pk+1} \leq 0$. Let us first note that:

$$p = a_{hk} + s = a_{h-1k} + c_{hk} + s \leq h - 1 < h.$$

Let us suppose now that $c_{pk} \leq 0$. In this case by (1) we see that:

$$a_{hk} + s = a_{h-1k} + c_{hk} + s < a_{hk-1} \leq a_{p-1k} \leq p,$$

which contradicts the fact that $a_{hk} + s = p$. So we can say that $c_{pk} = 1$.

Let us suppose now that $c_{pk+1} = 1$. Then by (1) we get:

$$a_{pk+1} = p + 1 = a_{hk} + s + 1.$$

By Theorem 1 we see that $a_{hk} \geq a_{hk+1}$ and we also have $a_{hk} < a_{hk} + s + 1 = a_{pk+1}$. This implies that $a_{hk+1} < a_{pk+1}$, but $p < h$ and so there exists i with $p < i \leq h$ such that $c_{ik+1} < 0$. Let $i \leq h$ such that $c_{ik+1} < 0$ and $a_{ik+1} = a_{hk+1} \leq a_{hk}$. By hypothesis on k it must be $a_{ik+1} < p + c_{ik+1} + 1$. So, taken i' such that $c_{i'k+1} < 0$ and $c_{i'+1k+1} = \dots = c_{h-1k+1} = 0$, we see that $a_{i'k+1} \leq p$. Again, by hypothesis it must be $a_{i'k+1} < p + c_{i'k+1} + 1$. Iterating the procedure we see that, taken m such that $c_{mk+1} = 1$ and $c_{m+1k+1} \leq 0$, it must be $a_{mk+1} \leq p$, where by (1) $a_{mk+1} = m + 1$. However, $c_{pk+1} = 1$ and so $m \geq p$ and so this gives us a contradiction. \square

In a similar way it is possible to prove the following:

Proposition 8. *Let $q \in \mathbb{N}$ such that:*

$$\{(i, j) \in T \mid q + c_{ij} + 1 \leq a_{ij} \leq q\} \neq \emptyset$$

and let:

$$h = \max\{i \mid \exists (i, j) \in T, q + c_{ij} + 1 \leq b_{ij} \leq q\}.$$

Then $0 \leq q \leq b$ and $\#(X \cap C_q) = h + 1$.

5. MAIN THEOREM

In this section we give some conditions for an admissible matrix to be the Hilbert matrix of some reduced zero-dimensional schemes. If M is an admissible matrix of size (a, b) , it is always possible to associate to M a reduced zero-dimensional scheme Z in the following way. Let R_0, \dots, R_a and C_0, \dots, C_b be pairwise distinct arbitrary $(1, 0)$ and $(0, 1)$ -lines. Let $P_{ij} = R_i \cap C_j$ and let us consider the scheme:

$$X = \{P_{ij} \mid c_{ij} = 1\}.$$

By proceeding as in [1, Proposition 4.1] we see that X is an ACM zero-dimensional scheme and that:

$$(11) \quad \Delta M_X^{(i,j)} = \begin{cases} 1 & \text{if } (i, j) \in X \\ 0 & \text{if } (i, j) \notin X. \end{cases}$$

Note that $(a_{ij} + r, b_{ij} + r) \in X$ for any $(i, j) \in T$ and $r \in I_{ij}$ (see (2) and (3)). Then it is easy to see that:

$$\mathcal{P} = \{P_{a_{ij}+r, b_{ij}+r} \mid (i, j) \in T, r \in I_{ij}\} \subsetneq X.$$

Definition 7. The scheme $Z = X \setminus \mathcal{P}$ is called *zero-dimensional scheme associated to M* .

We call Z the We want to show under which conditions the Hilbert matrix of Z is M . For this purpose we give the following definitions:

Definition 8. Let M be an admissible matrix. We say that M is a Δ -regular matrix if for any $(i_1, j_1), \dots, (i_n, j_n) \in T$ and $r_1 \in I_{i_1 j_1}, \dots, r_n \in I_{i_n j_n}$ the following conditions hold:

- (1) if $a_{i_1 j_1} + r_1 = \dots = a_{i_n j_n} + r_n$, $i_1 \neq \dots \neq i_n$ and $j_1 < \dots < j_n$, then $b_{i_1 j_1} + r_1 \leq \dots \leq b_{i_n j_n} + r_n$;
- (2) if $b_{i_1 j_1} + r_1 = \dots = b_{i_n j_n} + r_n$, $j_1 \neq \dots \neq j_n$ and $i_1 < \dots < i_n$, then $a_{i_1 j_1} + r_1 \leq \dots \leq a_{i_n j_n} + r_n$.

Remark 3. Given an admissible matrix M and any $(i_1, j_1), \dots, (i_n, j_n) \in T$ and $r_1 \in I_{i_1 j_1}, \dots, r_n \in I_{i_n j_n}$ such that $a_{i_1 j_1} + r_1 = \dots = a_{i_n j_n} + r_n$, $i_1 = \dots = i_n$ and $j_1 < \dots < j_n$, then by Proposition 6 it must be $b_{i_1 j_1} + r_1 > \dots > b_{i_n j_n} + r_n$.

Similarly, if $b_{i_1 j_1} + r_1 = \dots = b_{i_n j_n} + r_n$, $j_1 = \dots = j_n$ and $i_1 < \dots < i_n$, then by Proposition 5 $a_{i_1 j_1} + r_1 > \dots > a_{i_n j_n} + r_n$.

Definition 9. An admissible matrix M is called *plain matrix* if for any $(i_1, j_1), (i_2, j_2) \in T$, $r_1 \in I_{i_1 j_1}, r_2 \in I_{i_2 j_2}$ we have $(a_{i_1 j_1} + r_1, b_{i_1 j_1} + r_1) \neq (a_{i_2 j_2} + r_2, b_{i_2 j_2} + r_2)$.

Note that, if M is plain, then for any $(i_1, j_1), (i_2, j_2) \in T$, $r_1 \in I_{i_1 j_1}$ and $r_2 \in I_{i_2 j_2}$ we have $P_{a_{i_1 j_1} + r_1, b_{i_1 j_1} + r_1} \neq P_{a_{i_2 j_2} + r_2, b_{i_2 j_2} + r_2}$.

Example 1. Let us consider the following admissible matrix M and its first difference ΔM .

	0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	7	...
1	2	4	6	8	10	11	11	11	...
2	3	6	9	12	12	12	12	12	...
3	4	8	12	12	12	12	12	12	...
4	4	8	12	12	12	12	12	12	...
5

 M

	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	0	...
1	1	1	1	1	1	0	-1	0	...
2	1	1	1	1	-2	-1	0	0	...
3	1	1	1	-3	0	0	0	0	...
4	0	0	0	0	0	0	0	0	...
5

 ΔM

It is possible to see that M is Δ -regular and plain. Indeed, note that $T = \{(1, 6), (2, 5), (2, 4), (3, 3)\}$ and that:

- $c_{16} = -1$, so that $r = 0$ and $(a_{16}, b_{16}) = (0, 4)$;
- $c_{25} = -1$, so that $r = 0$ and $(a_{25}, b_{25}) = (0, 1)$;
- $c_{24} = -2$, so that $r = 0, 1$, $(a_{24}, b_{24}) = (0, 2)$ and $(a_{24} + 1, b_{24} + 1) = (1, 3)$;
- $c_{33} = -3$, so that $r = 0, 1, 2$, $(a_{33}, b_{33}) = (0, 0)$ and $(a_{33} + 1, b_{33} + 1) = (1, 1)$ and $(a_{33} + 2, b_{33} + 2) = (2, 2)$.

Since all these pairs are distinct, M is plain. It is Δ -regular because:

- taken $(3, 3), (2, 5), (1, 6) \in T$, we get $a_{33} = a_{25} = a_{16} = 0$ and $b_{33} = 0 < b_{25} = 1 < b_{16} = 4$;
- taken $(3, 3), (2, 4), (1, 6) \in T$, we get $a_{33} = a_{24} = a_{16} = 0$ and $b_{33} = 0 < b_{24} = 2 < b_{16} = 4$;

- taken $(3, 3), (2, 4) \in T$, we get $a_{33} + 1 = a_{24} + 1 = 1$ and $b_{33} + 1 = 1 < b_{24} + 1 = 3$;
- taken $(2, 5), (3, 3) \in T$, we get $b_{25} = b_{33} + 1 = 1$ and $a_{25} = 0 < a_{33} + 1 = 1$;
- taken $(2, 4), (3, 3) \in T$, we get $b_{24} = b_{33} + 2 = 2$ and $a_{24} = 0 < a_{33} + 2 = 2$.

Recalling Definition 7, we prove the following:

Theorem 6. *Let M be a plain and Δ -regular matrix such that one of the following conditions holds:*

- (1) $a_{ij} \geq a_{i-1j+1}$ for any $i, j \geq 0$;
- (2) $b_{ij} \geq b_{i+1j-1}$ for any $i, j \geq 0$.

Then $M^{(i,j)} = M_Z^{(i,j)}$ for any (i, j) .

Proof. Let us suppose that $b_{ij} \geq b_{i+1j-1}$ for any $i, j \geq 0$. Under this hypothesis we have that for any $(i_1, j), (i_2, j) \in T$ with $i_1 > i_2$ and for any $r_1 \in I_{i_1j}$ and $r_2 \in I_{i_2j}$ it is $b_{i_2j} + r_2 > b_{i_1j} + r_1$. Indeed, it is sufficient to show that $b_{i_2j} > b_{i_1j} - c_{i_1j} - 1 = b_{i_1j-1} - 1$. By hypothesis and by the fact that M is admissible we have:

$$b_{i_2j} \geq b_{i_2+1j-1} \geq b_{i_1j-1} > b_{i_1j-1} - 1.$$

Now we will prove that $\Delta M_Z^{(i,j)} = \Delta M^{(i,j)}$ for any (i, j) . We apply Corollary 1 by deleting one by one the points $(a_{ij} + r, b_{ij} + r)$, that are all distinct since M is plain. We proceed in the following way: given $(a_{i_1j_1} + r_1, b_{i_1j_1} + r_1)$ and $(a_{i_2j_2} + r_2, b_{i_2j_2} + r_2)$, we delete first $(a_{i_2j_2} + r_2, b_{i_2j_2} + r_2)$ if either $a_{i_1j_1} + r_1 < a_{i_2j_2} + r_2$ or $a_{i_1j_1} + r_1 = a_{i_2j_2} + r_2$ and $b_{i_1j_1} + r_1 < b_{i_2j_2} + r_2$.

Let us first show that it is possible to compute M_Z by applying recursively Corollary 1. Given the point $(a_{ij} + r, b_{ij} + r)$, with $c_{ij} < 0$ and $r \in I_{ij}$, by what we have just proved and by the fact that M is Δ -regular we see that:

$$\{(h, j) \mid b_{hj} + s = b_{ij} + r, h > i, s \in I_{hj}\} = \emptyset$$

and

$$(12) \quad \min\{h \mid (h, k) \in T, b_{hk} + s = b_{ij} + r, a_{hk} + s \geq a_{ij} + r, s \in I_{hk}\} = i.$$

So, keeping the notation of Corollary 1 and (12) together with Remark 2 and Proposition 8 imply:

$$(13) \quad q = \#(X \cap C_{b_{ij}+r}) - \#\{(h, k) \in T \mid b_{hk} + s = b_{ij} + r, a_{hk} + s \geq a_{ij} + r, s \in I_{hk}\} = i.$$

Let:

$$\begin{aligned} m_{ij}^{(r)} &= \min\{k \mid \exists (i, k) \in T, k \leq j, a_{ik} + s = a_{ij} + r, s \in I_{ik}\}, \\ n_{ij}^{(r)} &= \max\{k \mid \exists (i, k) \in T, k \geq j, a_{ik} + s = a_{ij} + r, s \in I_{ik}\}, \\ p_{ij}^{(r)} &= m_{ij}^{(r)} + \#\{(i, k) \mid a_{ik} + s = a_{ij} + r, k > j, s \in I_{ik}\}. \end{aligned}$$

Note that by Remark 2 and by Proposition 6:

$$(14) \quad p_{ij}^{(r)} = m_{ij}^{(r)} + n_{ij}^{(r)} - j.$$

By the fact that M is Δ -regular, by Remark 2 and by Propositions 6 and 7:

$$\#(X \cap R_{a_{ij}+r}) - \#\{(h, k) \in T \mid a_{hk} + s = a_{ij} + r, b_{hk} + s \geq b_{ij} + r, s \in I_{hk}\} = p_{ij}^{(r)}$$

that, in the notation of Corollary 1 and together with (13), gives that:

$$(15) \quad (q, p) = (i, p_{ij}^{(r)}).$$

Suppose that the first point to be deleted is $(a_{i_1 j_1} + r_1, b_{i_1 j_1} + r_1)$ and let $X' = X \setminus \{(a_{i_1 j_1} + r_1, b_{i_1 j_1} + r_1)\}$. Then by the fact that X is ACM we can apply Corollary 2:

$$\Delta M_{X'}^{(i,j)} = \begin{cases} \Delta M_X^{(i,j)} & \text{for } (i,j) \neq (i_1, p_{i_1 j_1}^{(r_1)}) \\ \Delta M_X^{(i,j)} - 1 & \text{for } (i,j) = (i_1, p_{i_1 j_1}^{(r_1)}). \end{cases}$$

Iterating the procedure, taken a point $(a_{i_1 j_1} + r_1, b_{i_1 j_1} + r_1)$, let us consider:

$$G = \{(a_{ij} + r, b_{ij} + r) \in \mathcal{P} \mid a_{ij} + r > a_{i_1 j_1} + r_1\} \cup \\ \cup \{(a_{ij} + r, b_{ij} + r) \in \mathcal{P} \mid a_{ij} + r = a_{i_1 j_1} + r_1, b_{ij} + r > b_{i_1 j_1} + r_1\}$$

and the correspondent set:

$$H = \{(i, p_{ij}^{(r)}) \mid (a_{ij} + r, b_{ij} + r) \in G\}.$$

If $X'' = X \setminus G$, suppose that we can apply Corollary 1 to the scheme X'' by deleting one by one all the points $(a_{ij} + r, b_{ij} + r) \in G$. In this way we see that:

$$(16) \quad \Delta M_{X''}^{(i,j)} = \begin{cases} \Delta M_X^{(i,j)} & \text{if } (i,j) \notin H \\ \Delta M_X^{(i,j)} - 1 & \text{if } (i,j) \in H. \end{cases}$$

We will show that we can apply Corollary 1 to scheme $X''' = X'' \setminus \{(a_{i_1 j_1} + r_1, b_{i_1 j_1} + r_1)\}$.

By (11), (15) and by (16) we know $\Delta M_{X''}^{(i,j)} < 0$ if and only if $(i,j) \in H$. By (15) we cannot apply Corollary 1 to X''' if $(i_1 + 1, p_{i_1 j_1}^{(r_1)} + 1) \leq (i_2, p_{i_2 j_2}^{(r_2)})$ for some $(i_2, p_{i_2 j_2}^{(r_2)}) \in H$. Since $m_{ij}^{(r)} \leq p_{ij}^{(r)} \leq n_{ij}^{(r)}$ for every $(i,j) \in T$, by Remark 2 we have that $(i_1, p_{i_1 j_1}^{(r_1)}), (i_2, p_{i_2 j_2}^{(r_2)}) \in T$ and that:

$$a_{i_1 p_{i_1 j_1}^{(r_1)}} + s_1 = a_{i_1 j_1} + r_1 \quad \text{and} \quad a_{i_2 p_{i_2 j_2}^{(r_2)}} + s_2 = a_{i_2 j_2} + r_2,$$

for some $s_1 \in I_{i_1 p_{i_1 j_1}^{(r_1)}}$ and $s_2 \in I_{i_2 p_{i_2 j_2}^{(r_2)}}$. This means that:

$$a_{i_1 p_{i_1 j_1}^{(r_1)}} + s_1 = a_{i_1 j_1} + r_1 \leq a_{i_2 j_2} + r_2 = a_{i_2 p_{i_2 j_2}^{(r_2)}} + s_2$$

where $i_1 < i_2$ and $p_{i_1 j_1}^{(r_1)} < p_{i_2 j_2}^{(r_2)}$, which contradicts Proposition 1. So we can apply Corollary 1 and we see that:

$$\Delta M_{X'''}^{(i,j)} = \begin{cases} \Delta M_{X''}^{(i,j)} & \text{for } (i,j) \neq (i_1, p_{i_1 j_1}^{(r_1)}) \\ \Delta M_{X''}^{(i,j)} - 1 & \text{for } (i,j) = (i_1, p_{i_1 j_1}^{(r_1)}). \end{cases}$$

By iterating the procedure we are able to compute M_Z .

Now, note that, taken $(i_1, p_{i_1 j_1}^{(r_1)})$ and taken $s_1 \in I_{i_1 p_{i_1 j_1}^{(r_1)}}$ such that $a_{i_1 p_{i_1 j_1}^{(r_1)}} + s_1 = a_{i_1 j_1} + r_1$, it is easy to see that:

$$m_{i_1 p_{i_1 j_1}^{(r_1)}}^{(s_1)} = m_{i_1 j_1}^{(r_1)} \quad \text{and} \quad n_{i_1 p_{i_1 j_1}^{(r_1)}}^{(s_1)} = n_{i_1 j_1}^{(r_1)}.$$

This implies together with (14) that $p_{p_{i_1 j_1}^{(r_1)}}^{(s_1)} = m_{i_1 p_{i_1 j_1}^{(r_1)}}^{(s_1)} + n_{i_1 p_{i_1 j_1}^{(r_1)}}^{(s_1)} - p_{i_1 j_1}^{(r_1)} = j$. This means that $\Delta M_Z^{(i,j)} = \Delta M^{(i,j)}$ for any (i,j) .

The proof works in a similar way if $a_{ij} \geq a_{i-1 j+1}$ for any $i, j \geq 0$. \square

Corollary 4. *Let M be an admissible matrix such that:*

$$a_{i_1 j_1} - b_{i_1 j_1} < a_{i_2 j_2} - b_{i_2 j_2}$$

for any $(i_1, j_1), (i_2, j_2) \in T$, with $i_1 < i_2$ and $j_1 > j_2$. Suppose that one of the following conditions holds:

- (1) $a_{ij} \geq a_{i-1, j+1}$ for any $i, j \geq 0$;
- (2) $b_{ij} \geq b_{i+1, j-1}$ for any $i, j \geq 0$.

Then $M^{(i, j)} = M_Z^{(i, j)}$ for any (i, j) .

Proof. If $a_{i_1 j_1} - b_{i_1 j_1} < a_{i_2 j_2} - b_{i_2 j_2}$ for any $(i_1, j_1), (i_2, j_2) \in T$, with $i_1 < i_2$ and $j_1 > j_2$, then M is plain and Δ -regular. Then the statement follows by Theorem 6. \square

Corollary 5. *Let M be a plain matrix and let $T = \{(i_1, j_1), \dots, (i_n, j_n)\}$. If $i_1 + j_1 = \dots = i_n + j_n$, then $M^{(i, j)} = M_Z^{(i, j)}$ for any (i, j) .*

Proof. We want to prove that M satisfies the hypothesis of Theorem 6.

By Proposition 3, Proposition 4 and by hypothesis for any i there exists at most one $j \in \mathbb{N}$ such that $c_{ij} < 0$ and, similarly, for any j there exists at most one $i \in \mathbb{N}$ such that $c_{ij} < 0$. Moreover, if $(i, j), (i+k, j-k) \in T$ for some $i, j, k \in \mathbb{N}$, then $(i+1, j-1), \dots, (i+k-1, j-k+1) \in T$.

Now we show that $a_{ij} \geq a_{i-1, j+1}$ for any i, j . If $c_{ij} = 1$, then this is true because M is an admissible matrix and $a_{ij} = \sum_{k \leq i} c_{kj}$. If $c_{ij} = 0$, by the fact that M is admissible:

$$a_{ij} = a_{i-1, j} \geq a_{i-1, j+1}.$$

If $c_{ij} < 0$, then $c_{ij+1} = 0$ and by the fact that M is admissible:

$$a_{ij} \geq a_{ij+1} = a_{i-1, j+1} + c_{ij+1} = a_{i-1, j+1}.$$

In a similar way it is possible to see that $b_{ij} \geq b_{i+1, j-1}$.

Now we need to prove that M is Δ -regular. It is sufficient to show that for any $(i, j), (i+1, j-1) \in T$:

$$b_{i+1, j-1} - a_{i+1, j-1} \leq b_{ij} - a_{ij}.$$

This holds because we have just proved that $b_{i+1, j-1} \leq b_{ij}$ and $a_{i+1, j-1} \geq a_{ij}$. \square

In the following example we give an application of Theorem 6.

Example 2. Given the following matrix M , it is easy to see that it satisfies the hypotheses of Theorem 6 and that its first difference ΔM is the following:

	0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	8	...
1	2	4	6	8	10	12	13	14	14	...
2	3	6	9	12	15	18	19	19	19	...
3	4	8	12	16	19	22	22	22	22	...
4	5	10	15	20	23	24	24	24	24	...
5	6	12	18	23	24	24	24	24	24	...
6	7	14	21	23	24	24	24	24	24	...
7	8	16	21	23	24	24	24	24	24	...
8	9	18	21	23	24	24	24	24	24	...
9	9	18	21	23	24	24	24	24	24	...
10

 M

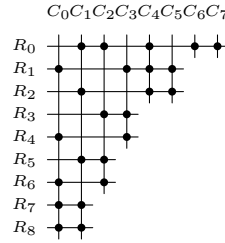
	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	0	...
1	1	1	1	1	1	1	0	0	0	...
2	1	1	1	1	1	1	0	-1	0	...
3	1	1	1	1	0	0	-1	0	0	...
4	1	1	1	1	0	-2	0	0	0	...
5	1	1	1	0	-2	-1	0	0	0	...
6	1	1	1	-3	0	0	0	0	0	...
7	1	1	-2	0	0	0	0	0	0	...
8	1	1	-2	0	0	0	0	0	0	...
9	0	0	0	0	0	0	0	0	0	...
10

 ΔM

We see that:

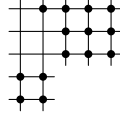
- $c_{27} = -1$, $a_{27} = 0$ and $b_{27} = 5$ and we get the point P_{05} ;
- $c_{36} = -1$, $a_{36} = 0$, and $b_{36} = 3$ and we get the point P_{03} ;
- $c_{45} = -2$, $a_{45} = 1$ and $b_{45} = 2$ and we get the points P_{12} and P_{23} ;
- $c_{55} = -1$, $a_{55} = 0$ and $b_{55} = 0$ and we get the point P_{00} ;
- $c_{54} = -2$, $a_{54} = 1$ and $b_{54} = 1$ and we get the points P_{11} and P_{22} ;
- $c_{63} = -3$, $a_{63} = 2$ and $b_{63} = 0$ and we get the points P_{20}, P_{31}, P_{42} ;
- $c_{72} = -2$, $a_{72} = 5$ and $b_{72} = 0$ and we get the points P_{50} and P_{61} ;
- $c_{82} = -2$, $a_{82} = 3$ and $b_{82} = 0$ and we get the points P_{30} and P_{41} .

By Theorem 6 we have that M is the Hilbert matrix of a scheme Z whose points can be represented in a grid of $(1, 0)$ and $(0, 1)$ -lines in the following way:

The scheme Z

Example 3. In this example we make some remarks on the hypotheses of Theorem 6.

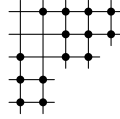
- (1) Let M be a plain matrix such that either condition 1 or condition 2 of Theorem 6 holds and suppose that it is not Δ -regular. Then it might be $M_Z \neq M$. As an example let us consider a scheme Y whose points can be represented in a grid of $(1, 0)$ and $(0, 1)$ -lines in the following way and the associated Hilbert matrix $M = M_Y$ which satisfies the previous conditions:

 Y

	0	1	2	3	4	5	6
0	1	2	3	4	5	5	...
1	2	4	6	8	10	10	...
2	3	6	9	12	14	14	...
3	4	8	11	13	14	14	...
4	5	10	13	14	14	14	...
5	5	10	13	14	14	14	...
6

 $M = M_Y$

Indeed, we can take, as in the definition of Δ -regular matrix, $(i_1, j_1) = (4, 3)$ and $i_2, j_2 = (3, 4)$. So $c_{43} = c_{34} = -1$, $a_{43} = a_{34} = 1$, while $b_{43} = 1$ and $b_{34} = 0$. This means that $b_{43} - b_{34} = 1 > 0 = a_{43} - a_{34}$. Then, by adding the points on the $(1, 0)$ -lines and using [1, Theorem 3.1], it is possible to see that $M_Z \neq M$:

 Z

	0	1	2	3	4	5	6
0	1	2	3	4	5	5	...
1	2	4	6	8	10	10	...
2	3	6	9	12	14	14	...
3	4	8	11	14	14	14	...
4	5	10	13	14	14	14	...
5	5	10	13	14	14	14	...
6

 M_Z

- (2) It is easy to see that the Hilbert matrix of 3 generic points of $\mathbb{P}^1 \times \mathbb{P}^1$ is such that $a_{ij} \geq a_{i-1, j+1}$ and $b_{ij} \geq b_{i+1, j-1}$ for any $i, j \geq 0$ and that it is Δ -regular, but it is not plain:

	0	1	2	3
0	1	2	3	...
1	2	3	3	...
2	3	3	3	...
3

Indeed, $a_{12} = a_{21} = 0$ and $b_{12} = b_{21} = 0$. In this case it is clear that $M_Z \neq M$, because $\deg Z = 4 \neq 3$.

- (3) Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a reduced zero-dimensional scheme whose points can be represented on a grid of $(1, 0)$ and $(0, 1)$ -lines in the following way:

 X

	0	1	2	3	4	5
0	1	2	3	4	4	...
1	2	4	6	8	8	...
2	3	6	7	8	8	...
3	4	8	8	8	8	...
4	4	8	8	8	8	...
5

 M_X

Then it is easy to see that the Hilbert matrix M_X of X is plain and Δ -regular, but it does not satisfy either condition 1 or condition 2 of Theorem 6. Indeed, $a_{22} = 1 < 2 = a_{13}$ and $b_{22} = 1 < 2 = b_{31}$. However, in this case $Z = X$.

Open problem. Given an admissible matrix M , which is plain and Δ -regular, but which does not satisfy either condition 1 or condition 2, is M the Hilbert function of some zero-dimensional schemes? In particular, given the associated scheme Z , $M_Z = M$?

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